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Refractive index tensors in connection with problems of photon scattering

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Abstract. In terms of the refractive index tensors N, the operator solving a problem of reflection and refraction of electromagnetic waves on plane boundary between isotropic dielectrics is given. Separately, the behaviour of Beltrami fields on the boundary is considered. They are described by infinite sets of branches of the traceless indefinite tensors N which are generators of involutive Maxwell groups for photon–antiphoton meeting pairs. The connections of the traceless tensors N with Fresnel reflection and transmission coefficients, Stokes relations, Beltrami fields and reflectional Coxeter's groups are established.

1. Introduction

A refractive index N is one of the oldest concepts. Its meaning has repeatedly become more precise and gained new content. Index N originally appeared as a scalar coefficient in the formulation of the kinematical reflection law and variational principles. It was ascertained that N depends on frequency in dispersive media and characterizes velocities of harmonic waves. The effect of birefringence is described by two refractive indices. Practical use of complex-valued functions also predetermined complexification of N for the description of optical dichroism. The discovery of the transversality of light waves pointed to their vector nature and led to further steps to make the meaning of N more precise. As a basic tool in geometrical optics Hamilton used scalar refractive indices and introduced analogous functions in mechanics. In modern mechanics operator equations, quaternions, Hamilton operators are used. But for a long time in optics N was treated as a scalar although Maxwell's equations themselves pointed to the possibility of operator generalization. Index N gained new significance in the works of Lorentz, who checked Fresnel formulae from the viewpoint of electromagnetic theory. This important problem was investigated by him in his doctoral thesis (1875) [1]. The 'microscopic point of view' of the mechanism of light refraction substantiated by Lorentz furthered the understanding of molecular optics [2]. A wave which has reached a medium causes oscillations of the dipole moments of each atom (or molecule). These dipole oscillations cause a secondary radiation. This radiation is considered to be the result of the incident wave scattering. Specific features of such coherent scattering have been investigated in many works [2]. The way in which the possibility of introducing refractive index N appears to be the result of forward wave scattering is explained by optical theory [3]. The angular distribution of the scattered unpolarized radiation at low frequencies is described by the Thomson formula. Radiation intensities forward and backwards are equal [4]. This equality is violated for high-energy photons (Klein–Nishina formula [4]).

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The refractive index N of a medium is determined by the formula $N_{ab} = 1+2\pi k^{-2} f_{ab}(0)$, $a, b = 1, 2$ where n is total of particles per unit volume, k is a wavenumber of the incident wave, $f_{ab}(0)$ is a 2×2 matrix amplitude of forward scattering for each separate atom. The complex matrix N_{ab} describes the effects of birefringence and dichroism including those caused by external actions. Faraday, Kerr, Pockels and Cotton–Mouton effects, which allow us to use the refractive index in approximation of the given field, are among such effects. The effect of self-focusing is among them also. In [5], for media with dispersion and nonlinearity without time-lag, the analysis of soliton solutions of the nonlinear wave equation is carried out by the new inverse transform method. Inverse problems of optics of complex media (bianisotropic, moving, dispersive) have become highly relevant recently [6–8]. Solving such problems (non-scalar in consideration of photon spin) for linear media leads to the almost unexplored field of exact solutions of nonlinear tensor wave equations by the inverse transform method. The existence of polarized one-directed solitary waves is one of the typical features of such solutions. In covariant crystal optics all the tensors are treated as operators of the three-dimensional space [9]. Refractive indices N are also the second-rank tensors of the three-dimensional space and play a substantial role in inverse problems of optics [6–9]. They generate global transfer operators (evolution Cauchy operators) and can be found by solving the tensor dispersion equations [10]. From similar equations the impedance operators Γ and frequency operators Ω can be found which describe the field evolution in the presence of inhomogeneities. In the works cited above and in a number of other works we widely used a spectral method of representation of the evolution solutions with the help of projective operators (polarization projectors). For these operators general covariant expressions were derived in which a wave normal and material tensors are involved. The internal unity of the spectral approach guarantees that the quantity of the tensor nonlinear wave equations integrated by the spectral transform method [11] will increase. In the case of isotropic media with dielectric permittivity ε and magnetic permeability μ, the operators N are found by extracting the square root $N = (\varepsilon\mu I)^{1/2}$ where $I^2 = I$ is a transverse projective operator. An infinite set of branches of N corresponds to even and odd states of the photon pairs (homogeneous and evanescent, co-directed and oppositely directed, having the same velocities [9, 10]). Branches of N generate continuous Maxwell groups and semi-groups of operators which describe a motion of vectors of the electromagnetic field along the generalized helices (boson strings [12, 13], Lakshmanan 1979, Reiter 1980, Green *et al* 1987) with lead $\lambda_0/(\varepsilon\mu)^{1/2}$. These helices wind around the elliptical cylinders inserted in each other. The traceless branches of N are associated with the rotation and reflection isometries which characterize metric properties of wave surfaces. In the case of the tensor Ω, elements of the cylinders are parallel to the time axis. The cylinders themselves are portions of two-dimensional toroids. Symmetries of the tensors N, Ω and Γ require that the helices (right-handed and left-handed) form a set of closed lines being geodesic on the toroids for circular polarizations. These non-plane lines in the three-dimensional space have non-zero curvatures and torsions. In [10] (Barkovsky and Furs 1997) the dependence of curvatures, torsions and Darboux methods on initial conditions and distance was found by the method of mobile Frenet trihedral. There the question was one of the so-called wave-guided solutions [14] described by the operators N. By replacing the spatial coordinate with time all the conclusions of [10] can be applied also to oscillator solutions [14] described by the frequency tensors Ω. For even states the traces of the tensors N turn out to be equal to $\pm 2(\varepsilon\mu)^{1/2}$ and for odd states to be equal to zero. This indicates the existence of photon–antiphoton, or in other words soliton–antisoliton, pairs. For a number of known nonlinear wave equations such symmetries of their soliton solutions are directly connected with the geometry of Riemann spaces of constant curvature [11, 12]. In the case under consideration the two-dimensional surfaces of wavefronts and mirrors are immersed in the three-dimensional space and are characterized by their own metric tensors and isometries connected with phase normals $\boldsymbol{n}$ and vectors $\boldsymbol{S}$, $\boldsymbol{C}$. These vectors are involved in N, Ω and Γ. It is thus necessary to consider manifolds with indefinite metrics [11, 13, 15, 16]. Traceless branches of the operators N, Ω and Γ are indefinite. As is shown in [10], they describe superpositions of elliptically polarized meeting waves in an isotropic medium, i.e. manifolds of standing elliptical helices closed on themselves. At the internal description of fields in the wavefront subspaces, three similar indefinite 2×2 matrices (Pauli matrices) appear in the representation of N, Ω and Γ. The appropriate fronts are associated with Möbius (one-sided) surfaces unoriented in the three-dimensional space. For quantization of an electromagnetic field one has resort to the use of indefinite metrics. In [17], Heitler used involutive indefinite metric operators for this purpose (see also [18]).

Indefinite N, Ω and Γ describe colliding optical solitons being elliptically polarized meeting photon–antiphoton pairs. It is known that dispersion and nonlinearity are not always necessary for wave equations to have soliton solutions [19]. It is essential that in a number of cases a possibility remains to apply to nonlinear media the concept of refractive index depending one way or another on intensity of light waves. Nonlinear refractive indices are responsible for self-trapping phenomena, for giving rise to induced waveguides (light-induced lensing). In recent years the family of spatial and temporal optical solitons was extended. Side by side with $(1 + 1)$D solitons stable $(2 + 1)$D rays were experimentally observed [20]. The latter are not localized in the same plane and have spiral form [20] in space. Rays (beams) are combined from angular spectra of plane waves. It is undoubted that in this combining, angular momentum plays an important role [21, 22]. The problem of its conservation for families of the soliton interactions in three dimensions was specially discussed in recent papers [20]. It is not excluded that the spatial oscillations of Darboux vectors discovered in [10] for elliptically polarized photons could be used in explanation of $(2 + 1)$D-ray spatial configurations.

In modern theory of optical waveguides the Fresnel formulae are widely used and inhomogeneous media with various profiles of the scalar refractive index are considered [23]. Recently, in a series of works [24], the existence of plane electromagnetic waves with parallel electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ vectors in free space was discussed. These works highlighted the general absence of information about such fields (of the Beltrami type) in classic monographs on electrodynamics. As an example, the book [4] of Jackson was pointed out. Here we show how the derivation of Fresnel formulae given in that book for interfaces between non-conductive isotropic electromagnetics can be extended to include fields with $\boldsymbol{E} \parallel \boldsymbol{H}$ and how this derivation can be stated completely in terms of the tensors N, taking into account their even and odd branches. The problem of Beltrami fields on the interfaces of dielectrics was earlier considered in [9] (Barkovsky *et al* 1996) but without using the tensor indices N. In the main part of this paper, with the use of the evolution solutions we analyse the problem of reflection and refraction of electromagnetic waves at oblique incidence to find Fresnel reflection and transmission operators R and T (section 2). Two cases are considered: when the trace of the refractive index tensor N is not equal to zero (running waves) and when N is traceless (fields of the Beltrami type). In section 3 only normal incidence of waves is considered in supposition that in both half-spaces the solutions are described by traceless N. For this case it is established that the refractive index tensors N have special ‘triangular’ form and can be expressed in terms of Fresnel coefficients r, r', t, t'. Also, we discuss the problem of the light reversibility and connections of the operator solutions with Coxeter groups.

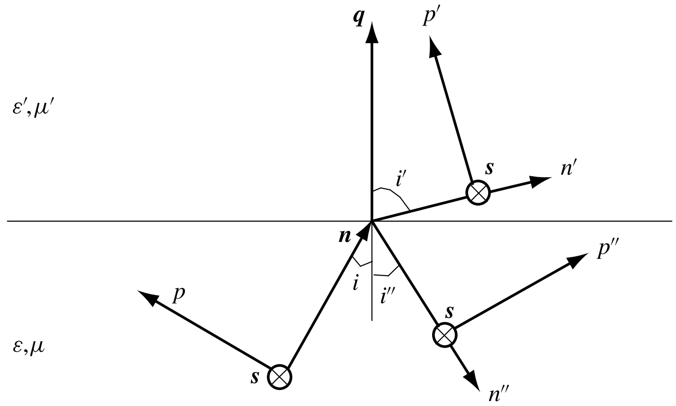

Figure 1. Disposition of wave normals and polarization vectors for incident, reflected and refracted waves.

2. Tensor N in the problem of coherent scattering on the plane boundary of electromagnetics and Beltrami fields

Let a plane electromagnetic wave with angular frequency ω and phase normal $\boldsymbol{n}$ be obliquely incident on the boundary between two semiinfinite homogeneous linear isotropic media from the direction of the lower medium having dielectric permittivity ε and magnetic permeability μ (figure 1). Accordingly, the upper medium has permittivity ε' and permeability μ'. The electromagnetic field of the wave is described by four vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ and its spatial variation is described by the scalar $\zeta = \boldsymbol{n} \cdot \boldsymbol{r}$ where $\boldsymbol{r}$ is the position vector of the observation point. For each value ζ there corresponds some fixed position of the wavefront which is perpendicular to $\boldsymbol{n}$. Supposing that the magnetic field intensity at $\zeta = 0$ is given, we can represent its value at another point by using the evolution formula

$$\boldsymbol{H}(\zeta) = \mathrm{e}^{\mathrm{i}k_0 N\zeta}\boldsymbol{H}(0) \tag{1}$$

where $k_0 = \omega/c$ is *vacuo* wavenumber and N is the second-rank tensor of refractive indices for the field $\boldsymbol{H}$. A covariant vector solution of the boundary problem as given by Fedorov (see, for example, [9] (Barkovsky and Borzdov 1997)). He used partial waves (partial solutions of the wave equation) which turn out to be effectively described by refraction vectors $\boldsymbol{m} = n\boldsymbol{n}$. In $\boldsymbol{m}$ a scalar n is the refractive index of the isotropic medium at frequency ω. The spatial part of the phase of such waves is expressed as $\varphi(\zeta) = \omega\boldsymbol{m} \cdot \boldsymbol{r}/c = k_0 n\boldsymbol{n} \cdot \boldsymbol{r} = k\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{k} \cdot \boldsymbol{r} = \omega\zeta/v$. Here $v = c/n$ is the phase velocity, $\boldsymbol{k}$ is the wavevector. Expression (1) is a generalization of partial solutions with scalar exponential factors. Now the argument of the exponential function in (1) contains not vector $\boldsymbol{m}$ but the third-rank tensor $\mathrm{i}k_0 N \otimes \boldsymbol{n}$ multiplied from the right by the position vector (or convoluted with $\boldsymbol{r}$ in the latter index: $(\mathrm{i}k_0 N \otimes \boldsymbol{n})\boldsymbol{r} = \mathrm{i}k_0 N\zeta = (\mathrm{i}k_0 N_{ij} n_k r_k)$, $i, j, k = 1, 2, 3$). As a result its rank decreases to two. The exponential operator in (1) acts on the vector $\boldsymbol{H}(0)$ and characterizes the variation of $\boldsymbol{H}$ at passage from the point $\zeta = 0$ to another point, $\zeta \neq 0$. In crystal optics there are four refractive index tensors $N^{(\mathrm{e})}$, $N^{(\mathrm{d})}$, $N^{(\mathrm{h})}$, $N^{(\mathrm{b})}$ describing spatial evolution of the field vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ respectively. In anisotropic media they differ from each other but in isotropic media they simply coincide.

The dispersion equation for finding N in an isotropic medium is

$$N^2 = -\boldsymbol{m}^{\times 2} = \boldsymbol{m}^2 - \boldsymbol{m} \otimes \boldsymbol{m} = \varepsilon\mu\boldsymbol{n}^{\times 2} \tag{2}$$

where $m_{il}^{\times} = e_{ikl}m_k$, e_{ikl} is the Levi-Civita pseudotensor, $i, j, k = 1, 2, 3$. Summation is meant by repeated indices. The antisymmetric pseudotensor of the second rank $\boldsymbol{m}^{\times}$ is dual to the vector $\boldsymbol{m}$. In a Cartesian basis this tensor can be considered as a 'matrix vector': $\boldsymbol{m}^{\times} = n\boldsymbol{n}^{\times} = m_1L_1 + m_2L_2 + m_3L_3$. Here m_i are usual Cartesian components of the vector $\boldsymbol{m}$, and antisymmetric '3×3 vectors' L_i of the matrix basis determine rotations with unit angular velocity about the coordinate axes. They are under the following commutation rules $[L_1, L_2] = L_3$, $[L_2, L_3] = L_1$, $[L_3, L_1] = L_2$. Basis spin matrices L_i (for spin 1) determine infinitesimal rotations of the space E_3. Operators of the type L_i were used by Helmholtz in his vortex theory. Appearance of the operator $\boldsymbol{n}^{\times}$ in optics is simply a consequence of Maxwell's equations containing the *curl* operator. Any tensor function of $\boldsymbol{n}^{\times}$ contains only summands with the tensors $\boldsymbol{n}^{\times}$ and $\boldsymbol{n}^{\times 2}$. The electromagnetic field of any wave is locally plane since its wavefront in a small neighbourhood may be replaced by the tangent plane with normal $\boldsymbol{n}$ which is immersed in the three-dimensional space.

We have already noted that there is an infinite set of solutions of (2). They are divided into two types: symmetric with non-zero trace and traceless ($\operatorname{tr} N = 0$). In the first case N are associated with co-directed waves and in the second with meeting waves. At first let us consider the symmetric branch

$$N = (\varepsilon\mu)^{1/2}(-\boldsymbol{n}^{\times 2}) = (\varepsilon\mu)^{1/2}I \qquad \operatorname{tr} N = 2(\varepsilon\mu)^{1/2}. \tag{3}$$

Operator I ($I^2 = I$) and diad $\boldsymbol{n} \otimes \boldsymbol{n}$ ($(\boldsymbol{n} \otimes \boldsymbol{n})^2 = \boldsymbol{n} \otimes \boldsymbol{n}$) are projective operators. The first relation in (3) is in essence the spectral expansion of the unit operator $1 = \boldsymbol{n} \otimes \boldsymbol{n} + I$ of the three-dimensional space into the one-dimensional subspace along $\boldsymbol{n}$ and two-dimensional subspace being a plane orthogonal to $\boldsymbol{n}$. Acting on any vector $\boldsymbol{a}$ the operator I projects it to this plane: $I\boldsymbol{a} = -\boldsymbol{n}^{\times 2}\boldsymbol{a} = -[\boldsymbol{n}[\boldsymbol{n}\boldsymbol{a}]] = \boldsymbol{a} - (\boldsymbol{n} \cdot \boldsymbol{a})\boldsymbol{n}$, and diad $\boldsymbol{n} \otimes \boldsymbol{n}$ projects $\boldsymbol{a}$ in the $\boldsymbol{n}$-direction (square brackets denote a vector product of two vectors). Now we choose on the wave surface of the incident wave (see figure 1) a pair of the unit mutually perpendicular vectors $\boldsymbol{s}$ and $\boldsymbol{p}$ which form with $\boldsymbol{n}$ the right-handed triad, the vector $\boldsymbol{s}$ being perpendicular ($\boldsymbol{s} \cdot \boldsymbol{n} = \boldsymbol{s} \cdot \boldsymbol{q} = 0$) to the incidence plane (the plane of figure 1). Then we can expand I

$$I = \boldsymbol{s} \otimes \boldsymbol{s} + \boldsymbol{p} \otimes \boldsymbol{p} \qquad [\boldsymbol{s}\boldsymbol{p}] = \boldsymbol{n} \tag{4}$$

into one-dimensional subspaces characterized by the symmetric projective diads $\boldsymbol{s} \otimes \boldsymbol{s}$ and $\boldsymbol{p} \otimes \boldsymbol{p}$ under conditions $\boldsymbol{s}^2 = \boldsymbol{p}^2 = 1$, $\boldsymbol{s} \cdot \boldsymbol{p} = \boldsymbol{s} \cdot \boldsymbol{n} = \boldsymbol{s} \cdot \boldsymbol{q} = 0$. It is clear that relation (4) is only one of the spectral expansions of I.

The phase normals $\boldsymbol{n}'$ and $\boldsymbol{n}''$ of the refracted and reflected waves are in the incidence plane whose orientation is given by vector $\boldsymbol{s}$. Therefore diad $\boldsymbol{s} \otimes \boldsymbol{s}$ is also involved in expressions for the transverse projectors of the refracted and reflected waves (see figure 1)

$$I' = -(\boldsymbol{n}'^{\times})^2 = 1 - \boldsymbol{n}' \otimes \boldsymbol{n}' = \boldsymbol{s} \otimes \boldsymbol{s} + \boldsymbol{p}' \otimes \boldsymbol{p}' \qquad [\boldsymbol{s}\boldsymbol{p}'] = \boldsymbol{n}' \tag{5}$$

$$I'' = -(\boldsymbol{n}''^{\times})^2 = 1 - \boldsymbol{n}'' \otimes \boldsymbol{n}'' = \boldsymbol{s} \otimes \boldsymbol{s} + \boldsymbol{p}'' \otimes \boldsymbol{p}'' \qquad [\boldsymbol{s}\boldsymbol{p}''] = \boldsymbol{n}''. \tag{6}$$

Since these waves run away from the boundary then triads $\boldsymbol{s}, \boldsymbol{p}', \boldsymbol{n}'$ and $\boldsymbol{s}, \boldsymbol{p}'', \boldsymbol{n}''$ together with $\boldsymbol{s}, \boldsymbol{p}, \boldsymbol{n}$ must be right-handed. In so doing, vectors $\boldsymbol{p}'$ and $\boldsymbol{p}''$ form angles of $i' - i$ and $\pi - 2i$, respectively, with vector $\boldsymbol{p}$. The same thing is valid for vectors $\boldsymbol{n}'$, $\boldsymbol{n}''$ and $\boldsymbol{n}$. We have

$$\boldsymbol{n}' = \mathrm{e}^{\boldsymbol{s}^{\times}(i'-i)}\boldsymbol{n} = [\boldsymbol{s}\boldsymbol{n}]\sin(i' - i) + \boldsymbol{n}\cos(i' - i)$$
$$\boldsymbol{n}'' = \mathrm{e}^{\boldsymbol{s}^{\times}(\pi-2i)}\boldsymbol{n} = [\boldsymbol{s}\boldsymbol{n}]\sin 2i - \boldsymbol{n}\cos 2i$$

where known representation of the rotation operator (versor) about the unit vector $\boldsymbol{s}$ through an angle φ is used

$$\exp(\boldsymbol{s}^{\times}\varphi) = \boldsymbol{s} \otimes \boldsymbol{s} + \boldsymbol{s}^{\times}\sin\varphi - \boldsymbol{s}^{\times 2}\cos\varphi.$$

Unit vectors $\boldsymbol{q}$, $\boldsymbol{n}$, $\boldsymbol{n}'$, $\boldsymbol{n}''$, $\boldsymbol{s}$, $\boldsymbol{p}$, $\boldsymbol{s}'$, $\boldsymbol{p}'$, $\boldsymbol{s}''$, $\boldsymbol{p}''$ specify orientations of ten two-sided (non-Möbius) planes in the three-dimensional space. Taking into consideration equations (3)–(6) we can write the refractive index tensors of the refracted and reflected waves as

$$N' = (\varepsilon'\mu')^{1/2} I' \qquad N'' = (\varepsilon\mu)^{1/2} I'' \qquad I'^2 = I' \qquad I''^2 = I''. \tag{7}$$

The difference of N'' from N is only in the propagation direction of the wave in the same medium. In contrast to material constants ε, μ, $n^2 = \varepsilon\mu$ the tensor N (and N', N'') is not tensor of material constants. The projection character of I, I', I'' (3), (7) enables us to clear tensors in exponents

$$\boldsymbol{H}(\zeta) = I\boldsymbol{H}(0)\mathrm{e}^{\mathrm{i}k\zeta} \qquad \boldsymbol{H}'(\zeta') = I'\boldsymbol{H}'(0)\mathrm{e}^{\mathrm{i}k'\zeta'} \qquad \boldsymbol{H}''(\zeta'') = I''\boldsymbol{H}''(0)\mathrm{e}^{\mathrm{i}k\zeta''}$$
$$k = k_0(\varepsilon\mu)^{1/2} \qquad k' = k_0(\varepsilon'\mu')^{1/2}. \tag{8}$$

Unit operators which are the first terms of expansions into a series of the tensor exponentials contain along with I, I', I'' the diads $\boldsymbol{n}\otimes\boldsymbol{n}$, $\boldsymbol{n}'\otimes\boldsymbol{n}'$, $\boldsymbol{n}''\otimes\boldsymbol{n}''$. These diads vanish in (8) in view of the field transversality. Acting on $\boldsymbol{H}(0)$, $\boldsymbol{H}'(0)$, $\boldsymbol{H}''(0)$ projectors I, I', I'' split them in the point $\zeta = \zeta' = \zeta'' = \boldsymbol{n}\cdot\boldsymbol{r} = \boldsymbol{n}'\cdot\boldsymbol{r} = \boldsymbol{n}''\cdot\boldsymbol{r} = 0$ into partial waves which propagate with equal velocities, for example, $(\varepsilon\mu)^{-1/2}N\boldsymbol{H}(0) = I\boldsymbol{H}(0) = (\boldsymbol{s}\otimes\boldsymbol{s} + \boldsymbol{p}\otimes\boldsymbol{p})\boldsymbol{H}(0) = \boldsymbol{s}(\boldsymbol{s}\cdot\boldsymbol{H}(0)) + \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{H}(0)) = \boldsymbol{H}_{\mathrm{s}}(0) + \boldsymbol{H}_{\mathrm{p}}(0)$. At steady state the phases of all the fields on the boundary must be equal. In the Cartesian coordinate system with origin on the boundary and with z-axis parallel to $\boldsymbol{q}$ we have

$$\boldsymbol{k}\cdot\boldsymbol{r}|_{\zeta=0} = \boldsymbol{k}'\cdot\boldsymbol{r}|_{\zeta=0} = \boldsymbol{k}''\cdot\boldsymbol{r}|_{\zeta=0} \qquad \boldsymbol{k}' = k'\boldsymbol{n}', \quad \boldsymbol{k}'' = k'\boldsymbol{n}''$$

whence equalities

$$(\varepsilon\mu)^{1/2}\sin i = (\varepsilon'\mu')^{1/2}\sin i' = (\varepsilon\mu)^{1/2}\sin i''$$

follow. They express kinematical reflection and refraction laws.

Dynamical reflection and refraction laws are described by Fresnel reflection and transmission operators R and T. For an arbitrary field $\boldsymbol{H}$ of the incident wave on the boundary we have by definition

$$\boldsymbol{H}' = T\boldsymbol{H} \qquad \boldsymbol{H}'' = R\boldsymbol{H} \tag{9}$$

where R and T characterize the fields of the reflected and transmitted waves on the boundary, respectively. For finding R and T in [25] a surface impedance tensor were used. Therefore in the formulae for R and T given in [25] the same for all waves tangential component $\boldsymbol{b}$ of the refraction vector and the normal vector $\boldsymbol{q}$ to the boundary are involved. We have

$$\boldsymbol{H}'_{\mathrm{s}} = (\boldsymbol{s}\otimes\boldsymbol{s})\boldsymbol{H}' = t_{\mathrm{s}}\boldsymbol{H}_{\mathrm{s}} = t_{\mathrm{s}}(\boldsymbol{s}\otimes\boldsymbol{s})\boldsymbol{H}$$
$$\boldsymbol{H}''_{\mathrm{s}} = (\boldsymbol{s}\otimes\boldsymbol{s})\boldsymbol{H}'' = r_{\mathrm{s}}\boldsymbol{H}_{\mathrm{s}} = r_{\mathrm{s}}(\boldsymbol{s}\otimes\boldsymbol{s})\boldsymbol{H}$$
$$\boldsymbol{H}'_{\mathrm{p}'} = (\boldsymbol{p}'\otimes\boldsymbol{p}')\boldsymbol{H}' = t_{\mathrm{p}}\boldsymbol{H}_{\mathrm{p}'} = t_{\mathrm{p}}(\boldsymbol{p}'\otimes\boldsymbol{p})\boldsymbol{H}$$
$$\boldsymbol{H}''_{\mathrm{p}''} = (\boldsymbol{p}''\otimes\boldsymbol{p}'')\boldsymbol{H}'' = r_{\mathrm{p}}\boldsymbol{H}_{\mathrm{p}''} = r_{\mathrm{p}}(\boldsymbol{p}''\otimes\boldsymbol{p})\boldsymbol{H}$$

where

$$t_{\mathrm{s}} = \frac{2}{1 + (\varepsilon\tan i/\varepsilon'\tan i')} \qquad r_{\mathrm{s}} = \frac{1 - (\varepsilon\tan i/\varepsilon'\tan i')}{1 + (\varepsilon\tan i/\varepsilon'\tan i')}$$
$$t_{\mathrm{p}} = 2\sqrt{\frac{\varepsilon\mu}{\varepsilon'\mu'}}\,\frac{\sin 2i}{(\varepsilon/\varepsilon')\sin 2i + \sin 2i'} \qquad r_{\mathrm{p}} = \frac{(\varepsilon/\varepsilon')\sin 2i - \sin 2i'}{(\varepsilon/\varepsilon')\sin 2i + \sin 2i'}. \tag{10}$$

In the same parametrization which have been used for representation of N now R and T take the form

$$R = r_{\mathrm{s}}\boldsymbol{s}\otimes\boldsymbol{s} + r_{\mathrm{p}}(\boldsymbol{n}\otimes\boldsymbol{p}\sin 2i - \boldsymbol{p}\otimes\boldsymbol{p}\cos 2i)$$
$$T = t_{\mathrm{s}}\boldsymbol{s}\otimes\boldsymbol{s} + t_{\mathrm{p}}(\boldsymbol{n}\otimes\boldsymbol{p}\sin(i' - i) + \boldsymbol{p}\otimes\boldsymbol{p}\cos(i' - i)).$$

Coefficients r_s, r_p, t_s, t_p characterize transverse anisotropy of reflection and transmission of a boundary of isotropic media which is the result of the oblique wave incidence. It is evident that replacements $T \to T + \boldsymbol{A} \otimes \boldsymbol{n}$, $R \to R + \boldsymbol{B} \otimes \boldsymbol{n}$ where $\boldsymbol{A}$ and $\boldsymbol{B}$ are arbitrary constant vectors do not change (9). This possibility follows from the gauge invariance of the electromagnetic field. At normal incidence we have

$$R = rI \qquad T = tI$$
$$I = 1 - \boldsymbol{n} \otimes \boldsymbol{n} = 1 - \boldsymbol{q} \otimes \boldsymbol{q} = 1 - \boldsymbol{n}' \otimes \boldsymbol{n}' = 1 - \boldsymbol{n}'' \otimes \boldsymbol{n}''$$

where

$$t = t_s = t_p = \frac{2}{1 + \sqrt{\varepsilon\mu'/\varepsilon'\mu}} \qquad r = r_s = -r_p = \frac{1 - \sqrt{\varepsilon\mu'/\varepsilon'\mu}}{1 + \sqrt{\varepsilon\mu'/\varepsilon'\mu}} \qquad t - r = 1. \tag{11}$$

Now we turn to other traceless (antisymmetric) solutions of the dispersion equation (2)

$$N_\pm = \pm \mathrm{i}\boldsymbol{m}^\times = \pm \boldsymbol{m} \cdot \boldsymbol{\sigma} = \pm(\varepsilon\mu)^{1/2}\boldsymbol{n} \cdot \boldsymbol{\sigma} \qquad \mathrm{tr}\, N_\pm = 0 \tag{12}$$

where $\sigma_k = \mathrm{i}L_k$, $k = 1, 2, 3$ is 1-spin 'matrix vector'. The operators $\pm\mathrm{i}\boldsymbol{m}^\times = \boldsymbol{m} \cdot \boldsymbol{\sigma}$ generate continuous group of rotations about the refraction vector $\boldsymbol{m}$. For homogeneous waves $\boldsymbol{m} = (\varepsilon\mu)^{1/2}\boldsymbol{n}$ with $\boldsymbol{n} = \boldsymbol{n}^*$ and $\boldsymbol{n}^2 = 1$. Tensor $\mathrm{i}\boldsymbol{n}^\times$ is the generator of the rotation group of electromagnetic field in vacuum ($\varepsilon = 1$, $\mu = 1$). Maxwell's equations for fields with angular frequency ω in an isotropic medium without charged sources have the form

$$\boldsymbol{\nabla}^\times \boldsymbol{H} = \mathrm{i}k_0\varepsilon\boldsymbol{E} \qquad \boldsymbol{\nabla}^\times \boldsymbol{E} = \mathrm{i}k_0\mu\boldsymbol{H} \qquad k_0 = \frac{\omega}{c} \tag{13}$$

where ε and μ do not depend on coordinates. Eliminating from (13) vector $\boldsymbol{E}$ we derive the Helmholtz equation for the field $\boldsymbol{H}$

$$\left(\boldsymbol{\nabla}^\times - k_0\sqrt{\varepsilon\mu}\right)\left(\boldsymbol{\nabla}^\times + k_0\sqrt{\varepsilon\mu}\right)\boldsymbol{H} = \left(\boldsymbol{\nabla}^\times + k_0\sqrt{\varepsilon\mu}\right)\left(\boldsymbol{\nabla}^\times - k_0\sqrt{\varepsilon\mu}\right)\boldsymbol{H} = 0. \tag{14}$$

Analogous equations can be derived for $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$. Alternative representation of (14) allows the equations

$$\left(\boldsymbol{\nabla}^\times + k_0\sqrt{\varepsilon\mu}\right)\boldsymbol{H} = 0 \qquad \left(\boldsymbol{\nabla}^\times - k_0\sqrt{\varepsilon\mu}\right)\boldsymbol{H} = 0. \tag{15}$$

Equations of type (15) are called Beltrami equations and corresponding fields are called Beltrami fields. In the case under consideration $\boldsymbol{\nabla}^\times \to \boldsymbol{n}^\times\partial/\partial\zeta$ for the field $\boldsymbol{H}(\zeta) = \boldsymbol{u}\exp(\mathrm{i}k\zeta)$, $k = k_0\sqrt{\varepsilon\mu}$ and instead of (15) we have

$$(\mathrm{i}\boldsymbol{n}^\times + 1)\boldsymbol{u} = 0 \qquad (\mathrm{i}\boldsymbol{n}^\times - 1)\boldsymbol{u} = 0.$$

Tensor $\mathrm{i}\boldsymbol{n}^\times$ has two complex circular (or isotropic) eigenvectors with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. An isotropic vector is perpendicular to itself. For the field $\boldsymbol{H}$ of the 'incident' wave (figure 1) these vectors have the form $\boldsymbol{u}_1 = (\boldsymbol{s} + \mathrm{i}\boldsymbol{p})/\sqrt{2}$, $\boldsymbol{u}_2 = \boldsymbol{u}_1^* = (\boldsymbol{s} - \mathrm{i}\boldsymbol{p})/\sqrt{2}$ and characterize circularly polarized waves of unit intensity. Since $\pm\mathrm{i} = \exp(\pm\mathrm{i}\pi/2)$ and $\boldsymbol{s} \cdot \boldsymbol{p} = 0$, $\boldsymbol{s}^2 = 1$, $\boldsymbol{p}^2 = 1$, $\boldsymbol{s} = \boldsymbol{s}^*$, $\boldsymbol{p} = \boldsymbol{p}^*$ then vector $\boldsymbol{u}_1$ is associated with right-handed polarized waves and vector $\boldsymbol{u}_2$ with left-handed polarized waves. The third eigenvector $\boldsymbol{u}_3 = \boldsymbol{n}$ corresponds to zero eigenvalue of the tensor $\mathrm{i}\boldsymbol{n}^\times$. Vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are isotropic inasmuch as $\boldsymbol{u}_1^2 = \boldsymbol{u}_1 \cdot \boldsymbol{u}_2^* = 0$, $\boldsymbol{u}_2^2 = \boldsymbol{u}_2 \cdot \boldsymbol{u}_1^* = 0$ and $\boldsymbol{u}_1 \cdot \boldsymbol{u}_1^* = 1$, $\boldsymbol{u}_2 \cdot \boldsymbol{u}_2^* = 1$, $\boldsymbol{n} \cdot \boldsymbol{u}_1 = \boldsymbol{n} \cdot \boldsymbol{u}_2 = 0$. Let us construct with the help of these vectors the Hermitian projective diads $\alpha = \boldsymbol{u}_1 \otimes \boldsymbol{u}_1^*$, $\alpha^* = \boldsymbol{u}_2 \otimes \boldsymbol{u}_2^*$. We obtain

$$\alpha = \alpha^+ = \alpha^2 = -\tfrac{1}{2}(\boldsymbol{n}^{\times 2} - \mathrm{i}\boldsymbol{n}^\times) \qquad \alpha^* = (\alpha^*)^+ = (\alpha^*)^2 = -\tfrac{1}{2}(\boldsymbol{n}^{\times 2} + \mathrm{i}\boldsymbol{n}^\times). \tag{16}$$

Here we have used the equalities $\boldsymbol{s}\otimes\boldsymbol{s}+\boldsymbol{p}\otimes\boldsymbol{p} = I = -\boldsymbol{n}^{\times 2} = 1-\boldsymbol{n}\otimes\boldsymbol{n}$ and $\boldsymbol{p}\otimes\boldsymbol{s}-\boldsymbol{s}\otimes\boldsymbol{p} = \boldsymbol{n}^\times$. Now the Hermitian tensor $\mathrm{i}\boldsymbol{n}^\times = \boldsymbol{n} \cdot \boldsymbol{\sigma}$ can be represented in the following spectral form

$$\mathrm{i}\boldsymbol{n}^\times = \boldsymbol{n} \cdot \boldsymbol{\sigma} = \lambda_1\boldsymbol{u}_1 \otimes \boldsymbol{u}_1^* + \lambda_2\boldsymbol{u}_2 \otimes \boldsymbol{u}_2^* = \alpha - \alpha^*.$$

A complex plane (Argand space of two-dimensional vectors) and theory of complex-valued functions provide the fundamental method of investigation of spin particles and antiparticles in quantum electrodynamics. It is easy to see that multiplication of the vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ by scalars $\pm\mathrm{i} = \exp(\pm\mathrm{i}\pi/2)$ or by pseudotensors $\pm\boldsymbol{n}^\times$ causes identical action in the plane formed by $\boldsymbol{s}$, $\boldsymbol{p}$. This action is a rotation by an angle of $\pi/2$. But combined application of these operations being multiplication by the resulting operator $\mathrm{i}\boldsymbol{n}^\times$ leaves the vector $\boldsymbol{u}_1$ fixed and changes the vector $\boldsymbol{u}_2$ into its opposite: $\boldsymbol{u}_2 \to -\boldsymbol{u}_2$. This statement, however, has relative sense and depends on agreement about the rotation direction by an angle of $\pi/2$. The combined operator $\mathrm{i}\boldsymbol{n}^\times$ can signify rotation by an angle of $\pi/2 - \pi/2 = 0$ or $\pi/2 + \pi/2 = \pi$, which enables us to speak about the relative parity of photons [26]. The mentioned agreement determines an orientation of the wavefront surface which is tangent to vectors $\boldsymbol{p}$ and $\boldsymbol{s}$. In the case of circular vectors which describe circularly polarized waves the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ yield to the equations

$$\boldsymbol{E} = \mp\mathrm{i}\boldsymbol{n}^\times\boldsymbol{E} \qquad \boldsymbol{H} = \mp\mathrm{i}\boldsymbol{n}^\times\boldsymbol{H} \qquad \boldsymbol{E} = \mp\mathrm{i}\sqrt{\frac{\mu}{\varepsilon}}\boldsymbol{H} \qquad \boldsymbol{E} = \mp\sqrt{\frac{\mu}{\varepsilon}}\boldsymbol{n}^\times\boldsymbol{H}.$$

From the latter equation it follows that for circularly polarized waves $\boldsymbol{E} \perp \boldsymbol{H}$ too.

According to Fresnel and Arago's observations, waves that are polarized mutually perpendicular do not interfere. Their velocities within the limits of the upper and lower subspaces coincide and are oppositely directed. What does the front of the resulting field represent? It is evident it will be plane and unoriented. Such surfaces are well known in geometry. They were first studied by Möbius.

Three diads α, α^* and $\boldsymbol{n}\otimes\boldsymbol{n}$ form a complete set of orthogonal projective operators of the three-dimensional space

$$\alpha + \alpha^* + \boldsymbol{n}\otimes\boldsymbol{n} = 1$$
$$\alpha\alpha^* = \alpha^*\alpha = \alpha(\boldsymbol{n}\otimes\boldsymbol{n}) = (\boldsymbol{n}\otimes\boldsymbol{n})\alpha = (\boldsymbol{n}\otimes\boldsymbol{n})\alpha^* = \alpha^*(\boldsymbol{n}\otimes\boldsymbol{n}) = 0.$$

For traceless operator N (12) the spatial evolution operator describes a rotation of the initial vector of the magnetic field intensity

$$\begin{aligned}\exp(\mathrm{i}k_0N\zeta) &= \exp(-\boldsymbol{k}^\times\zeta) = \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\sigma}\zeta)\\ &= \alpha\mathrm{e}^{-\mathrm{i}k\zeta} + \alpha^*\mathrm{e}^{\mathrm{i}k\zeta} = -\boldsymbol{n}^{\times 2}\cos k\zeta + \boldsymbol{n}^\times\sin k\zeta.\end{aligned} \tag{17}$$

In (17) we do not write a diad $\boldsymbol{n}\otimes\boldsymbol{n}$ for the reason noted above, when we wrote relations (8). Orthogonal projective operators α, α^* are simultaneously coherence tensors (polarization density matrices) of waves of unit intensity with right-handed and left-handed polarizations. If the matrix basis L_k is used, then instead of (16) one can write

$$\alpha = \tfrac{1}{2}[\boldsymbol{n}\cdot\boldsymbol{\sigma} + (\boldsymbol{n}\cdot\boldsymbol{\sigma})^2] \qquad \alpha^* = \tfrac{1}{2}[-\boldsymbol{n}\cdot\boldsymbol{\sigma} + (\boldsymbol{n}\cdot\boldsymbol{\sigma})^2] \tag{18}$$

where now the 1-spin matrix components of the 'vector' $\boldsymbol{\sigma}$ ($\sigma_k = \mathrm{i}L_k$) are used. The operator (17) splits a field in the initial point into two independent partial waves, right-handed and left-handed polarized, which propagate in opposite directions with the same velocities $c(\varepsilon\mu)^{-1/2}$. From (16)–(18) it follows that replacement $\boldsymbol{n} \to -\boldsymbol{n}$ leads to $\zeta \to -\zeta$, $\alpha \to \alpha^*$. Such symmetry shows the polarization reversibility of light waves in unbounded isotropic media.

The complex vector $\boldsymbol{H}(0)$ in (1) characterizes the field produced by some source in the initial point $\zeta = 0$. Any source has to obey some general requirements caused by the nature of the generated particles (waves). The operator formula (17) can be interpreted in terms of photons and antiphotons. In this formula the photon and antiphoton projectors $\alpha = \boldsymbol{u}_1\otimes\boldsymbol{u}_1^*$ and $\alpha^* = \boldsymbol{u}_1^*\otimes\boldsymbol{u}_1$ divide $\boldsymbol{H}(0)$ into two components: $\boldsymbol{H}(0) = (\alpha+\alpha^*)\boldsymbol{H}(0) = (\boldsymbol{u}_1^*\cdot\boldsymbol{H}(0))\boldsymbol{u}_1 + (\boldsymbol{u}_1\cdot\boldsymbol{H}(0))\boldsymbol{u}_1^*$. The squares $|\boldsymbol{u}_1^*\cdot\boldsymbol{H}|^2$ and $|\boldsymbol{u}_1\cdot\boldsymbol{H}|^2$ of these components determine the probabilities that photons have polarizations $\boldsymbol{u}_1$ and $\boldsymbol{u}_1^*$. The notion about a massless particle with certain helicity is invariant under orthochronous Lorentz transformations. Helical states always appear in pairs. This is a consequence of Maxwell's equations. It would be impossible to develop a theory if photons appeared in this theory, for instance, with positive helicity only. It is for the same reason that a theory with only positively charged particles is impossible. In such a theory the equivalence of all space–time points would be broken.

Returning to the boundary problem considered above, solution (12) evidently describes another situation, where one of the partial waves propagates to the boundary and the second wave does from the boundary in the opposite direction. Traces $\mathrm{tr}(\mathrm{i}\boldsymbol{n}^{\times}\alpha^{*}) = 1$ and $\mathrm{tr}(-\mathrm{i}\boldsymbol{n}^{\times}\alpha) = -1$ characterize the angular momenta of the right-handed polarized incident wave and the left-handed polarized wave propagating to meet the incident wave. The latter wave may be excited by the incidence of two waves with reversed normals, $-\boldsymbol{n}'$ and $-\boldsymbol{n}''$. It is clear that the branches $N = \pm\mathrm{i}\boldsymbol{m}^{\times}$ of the refractive index (12) point out the reversibility of polarized waves in an isotropic medium. The problem of mechanical and optical reversibility has been studied in detail in the works of Fresnel, Stokes, Helmholtz, Rayleigh, Ritz and Einstein. After discovery of radioactivity and spin such problems have arisen in many other dynamics. Since group theory methods and geometrical (topological) approaches are now widespread in modern physics, reversibility problems have become relevant in new branches of physics such as photonic crystals, chiral media, chemical waves, scattering by atoms in optical lattices, and chaos.

In [24] the problem of the existence of transverse waves with parallel vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ was discussed. The superposition of meeting partial waves, described by the operator (17), belongs to rotating light fields having the resulting energy flux vector equal to zero. In so doing instantaneous positions of the vectors $\boldsymbol{E} \parallel \boldsymbol{H}$ at different points ζ fall on helicoidal surfaces with right-handed and left-handed helices having positive and negative torsions, respectively. In differential geometry a positive sign is usually attributed to such torsion of a curve when rotation of its binormal is anticlockwise with respect to the observer. The curve itself is called 'right-handed'. Otherwise a negative sign is attributed and the curve is called 'left-handed'. Since, in the case under consideration, any sign of ζ is allowed then a right-handed circularly polarized wave with $\zeta = \boldsymbol{n}\cdot\boldsymbol{r} > 0$ is a left-handed polarized wave at negative $\zeta = \boldsymbol{n}\cdot\boldsymbol{r} < 0$.

It is easy to check that Fresnel coefficients (10) r_{s}, r_{p}, t_{s}, t_{p} for 'straight' and r'_{s}, r'_{p}, t'_{s}, t'_{p} for 'reversed' ($\boldsymbol{n} \to -\boldsymbol{n}, \boldsymbol{n}' \to -\boldsymbol{n}', \boldsymbol{n}'' \to -\boldsymbol{n}''$) waves are connected by Stokes relations

$$\begin{aligned} r'_{\mathrm{s}} &= -r_{\mathrm{s}} \qquad & r_{\mathrm{s}}^2 + t_{\mathrm{s}}t'_{\mathrm{s}} &= 1 \\ r'_{\mathrm{p}} &= -r_{\mathrm{p}} \qquad & r_{\mathrm{p}}^2 + t_{\mathrm{p}}t'_{\mathrm{p}} &= 1. \end{aligned} \tag{19}$$

At normal incidence with $r_{\mathrm{s}} = r_{\mathrm{p}} = r$ and $t_{\mathrm{s}} = t_{\mathrm{p}} = t$ we have the same relations. In [10] (Barkovsky and Fedorov 1993) the generalized tensor Stokes relations in optical bigyrotropic and bianisotropic systems were considered.

In terms of photons, phonons and other -ons the reversibility relations of type (19) express the law of P-parity conservation. From relations (11) for Fresnel coefficients in the case of normal incidence it follows, in particular, that starting from co-directed and meeting p- and s-polarized waves on the boundary one can obtain circularly and elliptically right-handed and left-handed polarized waves running to the boundary and away from the boundary.

3. Connections of traceless tensors N with Fresnel coefficients

In [10] we have considered the operator evolution solutions of Maxwell's equations for monochromatic (with time dependence $\sim\mathrm{e}^{-\mathrm{i}\omega t}$) electromagnetic plane waves in unbounded homogeneous isotropic media. It was shown that variation of an electromagnetic field in such media is locally characterized by the refractive index tensor N. The trace of this tensor can be equal to zero. In this case the field is a superposition of two meeting waves with polarizations $[\boldsymbol{nC}]$ and $\boldsymbol{S}$ which do not coincide

$$\boldsymbol{H}(z) = \alpha[\boldsymbol{nC}]\mathrm{e}^{\mathrm{i}kz} + \beta\boldsymbol{S}\mathrm{e}^{-\mathrm{i}kz}. \tag{20}$$

Vectors $\boldsymbol{S}$ and $\boldsymbol{C}$ are involved in the refractive index tensor $N = (\varepsilon\mu)^{1/2}(I - 2\boldsymbol{S}\otimes\boldsymbol{C})$ and are under conditions $\boldsymbol{S}\cdot\boldsymbol{C} = 1$, $\boldsymbol{S}\cdot\boldsymbol{n} = \boldsymbol{C}\cdot\boldsymbol{n} = 0$, $I = 1 - \boldsymbol{n}\otimes\boldsymbol{n}$, 1 is the unit tensor of the three-dimensional space, and the unit vector $\boldsymbol{n}$ determines the positive direction of the z-axis. The quantities α and β appearing in (20) are expansion coefficients of the initial field vector $\boldsymbol{H}(0)$ in basis $[\boldsymbol{nC}]$, $\boldsymbol{S}$:

$$\boldsymbol{H}(0) = \alpha[\boldsymbol{nC}] + \beta\boldsymbol{S}.$$

The evolution form of equation (20) is

$$\boldsymbol{H}(z) = \exp[\mathrm{i}kz(I - 2\boldsymbol{S}\otimes\boldsymbol{C})]\boldsymbol{H}(0) \tag{21}$$

where the exponential acts to the initial vector $\boldsymbol{H}(0)$ as a matrix factor. Also, there are other evolution solutions of Maxwell's equations of the type $\boldsymbol{H}(t) = \exp[-\mathrm{i}\omega t(I - 2\boldsymbol{S}\otimes\boldsymbol{C})]\boldsymbol{H}(0)$ when spatial dependence $\sim\mathrm{e}^{\mathrm{i}kz}$ of the field amplitude $\boldsymbol{H}$ is assumed to be given (so-called oscillator solutions). They are described by the frequency operator $\Omega = \omega(I - 2\boldsymbol{S}\otimes\boldsymbol{C})$ where $\boldsymbol{S}\cdot\boldsymbol{C} = 1$, $\boldsymbol{S}\cdot\boldsymbol{n} = \boldsymbol{C}\cdot\boldsymbol{n} = 0$ too. If we consider two particular solutions $N \sim \mathrm{i}\boldsymbol{n}^{\times}$ on the one hand and $\Omega \sim \mathrm{i}\boldsymbol{n}^{\times}$ on the other hand then it is easy to make sure by straightforward calculation that in both cases $\boldsymbol{E} \parallel \boldsymbol{H}$ and therefore the energy flux of such fields is identically equal to zero. These situations correspond to the examples of fields with $\boldsymbol{E} \parallel \boldsymbol{H}$ considered in [24] without the use of operator methods. But only for the first case ($N \sim \mathrm{i}\boldsymbol{n}^{\times}$) the *curl* of the field is proportional to the field itself (i.e. the field is of the Beltrami type).

The impedance tensor γ which connects the fields vectors $\boldsymbol{H}$ and $[\boldsymbol{nE}]$ as $[\boldsymbol{nE}] = \gamma\boldsymbol{H}$ can be calculated for waves of type (20). Taking into account one of the Maxwell equations $[\boldsymbol{n}\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}z}] = -\mathrm{i}\omega\varepsilon\boldsymbol{E}/c$ and equation (21) we conclude that

$$\gamma = Z(I - 2\boldsymbol{S}\otimes\boldsymbol{C})$$

where $Z = (\mu/\varepsilon)^{1/2}$.

Now let us turn to reflection and refraction of fields of type (20) on the interface of two isotropic media which are characterized by dielectric permittivities and magnetic permeabilities ε, μ ($z < 0$) and ε', μ' ($z > 0$), respectively. We shall suppose that the wave incidence is normal and $\boldsymbol{n}$ is the unit normal vector to the interface $z = 0$. Let the fields in the first and the second media be described by the traceless refractive index tensors $N = (\varepsilon\mu)^{1/2}(I - 2\boldsymbol{S}\otimes\boldsymbol{C})$ and $N' = (\varepsilon'\mu')^{1/2}(I - 2\boldsymbol{S}'\otimes\boldsymbol{C}')$ where

$$\boldsymbol{S}\cdot\boldsymbol{C} = \boldsymbol{S}'\cdot\boldsymbol{C}' = 1 \qquad \boldsymbol{S}\cdot\boldsymbol{n} = \boldsymbol{C}\cdot\boldsymbol{n} = \boldsymbol{S}'\cdot\boldsymbol{n} = \boldsymbol{C}'\cdot\boldsymbol{n} = 0. \tag{22}$$

It is evident that boundary conditions impose additional restrictions on the vectors $\boldsymbol{S}$, $\boldsymbol{C}$, $\boldsymbol{S}'$, $\boldsymbol{C}'$. From the relations

$$\boldsymbol{H}(0) = \boldsymbol{H}'(0) \qquad [\boldsymbol{nE}](0) = [\boldsymbol{nE}'](0) \tag{23}$$

and

$$[\boldsymbol{nE}] = \gamma\boldsymbol{H} \qquad [\boldsymbol{nE}'] = \gamma'\boldsymbol{H}'$$

it follows that

$$(\gamma - \gamma')\boldsymbol{H}(0) = 0 \tag{24}$$

i.e. the field vector $\boldsymbol{H}(0)$ on the interface has to be an eigenvector of the tensor $\gamma - \gamma'$ with zero eigenvalue. This is possible only when the condition

$$\det(\gamma - \gamma') = -\tfrac{1}{2}\,\mathrm{tr}(\gamma - \gamma')^2 = 0 \tag{25}$$

is fulfilled. Substituting in (25) the material impedance tensors $\gamma = Z(I - 2\boldsymbol{S} \otimes \boldsymbol{C})$ and $\gamma' = Z'(I - 2\boldsymbol{S}' \otimes \boldsymbol{C}')$ with $Z = (\mu/\varepsilon)^{1/2}$ and $Z' = (\mu'/\varepsilon')^{1/2}$ and taking into account relations (22) we receive after uncomplicated transformations

$$(\boldsymbol{S} \cdot \boldsymbol{C}')(\boldsymbol{S}' \cdot \boldsymbol{C}) = \frac{(Z + Z')^2}{4ZZ'} = \frac{1}{T} \tag{26}$$

where T is the energy transmission factor connected with the energy reflection factor R by the relation $R + T = 1$. The condition (26) is just the additional restriction imposed on the vectors $\boldsymbol{S}$, $\boldsymbol{C}$, $\boldsymbol{S}'$, $\boldsymbol{C}'$ because of the boundary conditions (23). If we specify vectors $\boldsymbol{S}$ and $\boldsymbol{C}$ for the first medium and initial vector $\boldsymbol{H}(0)$ on the interface then vectors $\boldsymbol{S}'$ and $\boldsymbol{C}'$ are not arbitrary and they have to be under conditions (26) and (24).

In problems of wave reflection and refraction as a rule three waves are under consideration. These waves are incident, reflected and refracted (or transmitted). Since the traceless refractive index tensors describe meeting waves then in this case it is necessary to consider two sets of waves (two incident, two reflected and two refracted waves) and each set has to be characterized by its own polarization. Let us specify two mutually orthogonal unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ on the interface ($\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{n} \cdot \boldsymbol{a} = \boldsymbol{n} \cdot \boldsymbol{b} = 0$, $[\boldsymbol{ab}] = \boldsymbol{n}$) and associate them with polarizations of each set of waves. We assume that from the direction of the first medium ($z < 0$) the $\boldsymbol{a}$-polarized wave of the unit intensity is incident on the interface, and from the direction of the second medium ($z > 0$) the $\boldsymbol{b}$-polarized of the unit intensity is incident. Then, in the first medium the reflected wave with polarization $r\boldsymbol{a}$ and the transmitted wave with polarization $t'\boldsymbol{b}$ are excited and in the second medium the reflected wave with polarization $r'\boldsymbol{b}$ and the transmitted wave with polarization $t\boldsymbol{a}$ are excited. Coefficients r, t, r', t' are Fresnel reflection and transmission coefficients and yield to the Stoke reversibility relations

$$r' = -r \qquad r^2 + tt' = 1.$$

For the boundary of two isotropic media

$$r = \frac{Z - Z'}{Z + Z'} \qquad t = \frac{2Z}{Z + Z'} \qquad r' = \frac{Z' - Z}{Z + Z'} \qquad t' = \frac{2Z'}{Z + Z'}. \tag{27}$$

The field distribution in the first medium will be described by the expression

$$\boldsymbol{H}(z) = \boldsymbol{a} \exp(\mathrm{i}kz) + (r\boldsymbol{a} + t'\boldsymbol{b}) \exp(-\mathrm{i}kz) \tag{28}$$

and in the second by

$$\boldsymbol{H}'(z) = (t\boldsymbol{a} + r'\boldsymbol{b}) \exp(\mathrm{i}k'z) + \boldsymbol{b} \exp(-\mathrm{i}k'z) \tag{29}$$

where $k = (\varepsilon\mu)^{1/2}\omega/c$, $k' = (\varepsilon'\mu')^{1/2}\omega/c$. On the interface

$$\boldsymbol{H}(0) = (1 + r)\boldsymbol{a} + t'\boldsymbol{b} \qquad \boldsymbol{H}'(0) = t\boldsymbol{a} + (1 + r')\boldsymbol{b} \tag{30}$$

and in view of (23),

$$1 + r = t \qquad 1 + r' = t'.$$

But the latter equalities are evident from (27).

We compare (28) and (29) with (20) and identify vectors $\boldsymbol{S}$, $\boldsymbol{C}$ and $\boldsymbol{S}'$, $\boldsymbol{C}'$ for each of the media. Without loss of generality for the first medium we can take

$$\boldsymbol{S} = r\boldsymbol{a} + t'\boldsymbol{b}. \tag{31}$$

The vector $\boldsymbol{C}$ has to be co-directed with the vector $\boldsymbol{b}$ (see (28) and (20)). Since $\boldsymbol{S} \cdot \boldsymbol{C} = 1$ then

$$\boldsymbol{C} = \frac{1}{t'}\boldsymbol{b}. \tag{32}$$

In this case the expansion coefficients of the vector $\boldsymbol{H}(0)$ in basis $[\boldsymbol{nC}]$, $\boldsymbol{S}$ are

$$\alpha = -t' \qquad \beta = 1.$$

We have

$$\boldsymbol{S} \otimes \boldsymbol{C} = \boldsymbol{b} \otimes \boldsymbol{b} + \frac{r}{t'}\boldsymbol{a} \otimes \boldsymbol{b}$$

and then the refractive index tensor N is

$$N = (\varepsilon\mu)^{1/2}(I - 2\boldsymbol{S} \otimes \boldsymbol{C}) = (\varepsilon\mu)^{1/2}\left(\boldsymbol{a} \otimes \boldsymbol{a} - \boldsymbol{b} \otimes \boldsymbol{b} - \frac{2r}{t'}\boldsymbol{a} \otimes \boldsymbol{b}\right).$$

Making the transition to the wavefront subspace and directing the x-axis and y-axis along $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, we now write N in the matrix form

$$N = (\varepsilon\mu)^{1/2}\begin{pmatrix} 1 & -2r/t' \\ 0 & -1 \end{pmatrix}. \tag{33}$$

Matrix (33) has non-orthogonal eigenvectors in view of its non-symmetric 'triangular' form. By similarity transformation this matrix can be reduced to the diagonal matrix with 1 and -1 on the diagonal. So the refractive index tensors under consideration represented as matrices are similar to Pauli matrices (Pauli matrices are similar to each other). Matrices of type (33) form a group. Lie algebra of this group and their representations are considered in [27].

For the second medium we take

$$\boldsymbol{C}' = -r'\boldsymbol{a} + t\boldsymbol{b} \tag{34}$$

and

$$\boldsymbol{S}' = \frac{1}{t}\boldsymbol{b}. \tag{35}$$

Then

$$\alpha' = -1 \qquad \beta' = t \qquad \boldsymbol{S}' \otimes \boldsymbol{C}' = \boldsymbol{b} \otimes \boldsymbol{b} - \frac{r'}{t}\boldsymbol{b} \otimes \boldsymbol{a}$$

and

$$\begin{aligned} N' &= (\varepsilon'\mu')^{1/2}(I - 2\boldsymbol{S}' \otimes \boldsymbol{C}') = (\varepsilon'\mu')^{1/2}\left(\boldsymbol{a} \otimes \boldsymbol{a} - \boldsymbol{b} \otimes \boldsymbol{b} + \frac{2r'}{t}\boldsymbol{b} \otimes \boldsymbol{a}\right) \\ N' &= (\varepsilon'\mu')^{1/2}\begin{pmatrix} 1 & 0 \\ 2r'/t & -1 \end{pmatrix}. \end{aligned} \tag{36}$$

Thus we have shown (see (33) and (36)) that in the case of reflection and refraction of meeting waves, non-diagonal matrix elements of the refractive index tensors are expressed through Fresnel reflection and refraction coefficients. It remains to verify the satisfiability of the early obtained condition (26). From (31), (32), (34), (35) it follows that

$$(\boldsymbol{S} \cdot \boldsymbol{C}')(\boldsymbol{S}' \cdot \boldsymbol{C}) = (-rr' + tt')\frac{1}{tt'} = \frac{1}{tt'}.$$

Substituting relations (27) in the latter expression we ensure that condition (26) is satisfied. Also one can verify by straightforward calculation that the vector $\boldsymbol{H}(0) = \boldsymbol{H}'(0)$ (30) is the eigenvector of the tensor $\gamma - \gamma'$ with zero eigenvalue.

We make another note concerning the operators $X = (\varepsilon\mu)^{-1/2}N = I - 2\boldsymbol{S} \otimes \boldsymbol{C}$. In an unbounded homogeneous isotropic medium only conditions of the form (22) are imposed on vectors $\boldsymbol{S}$ and $\boldsymbol{C}$, i.e. in the plane perpendicular to the phase normal $\boldsymbol{n}$ there is an infinite set of vectors $\boldsymbol{S}$ and $\boldsymbol{C}$ which satisfy (22). The set of pairs of complex vectors $\boldsymbol{S}$ and $\boldsymbol{C}$ in the evolution solutions corresponds to the infinite set of the three-dimensional generalized helices of the elliptical cylinders with elements parallel to $\boldsymbol{n}$. In the particular case of $\boldsymbol{S} = \boldsymbol{C} = \boldsymbol{S}^*$, $\boldsymbol{S} \cdot \boldsymbol{C} = \boldsymbol{S}^2 = 1$ the set of operators $X = I - 2\boldsymbol{S} \otimes \boldsymbol{S} = 1 - \boldsymbol{n} \otimes \boldsymbol{n} - 2\boldsymbol{S} \otimes \boldsymbol{S}$ with different orientations of $\boldsymbol{S}$ corresponds to an infinite set of plane mirrors containing the normal $\boldsymbol{n}$. Their orientations in the three-dimensional space are determined by vectors $\boldsymbol{S}$ perpendicular to $\boldsymbol{n}$, the replacement $\boldsymbol{S} \to -\boldsymbol{S}$ being not essential. In so doing tensor N describes the evolution of linear polarized waves when an initial vector is real. We have already noted earlier [10] that the group of operators X with discrete orientations of $\boldsymbol{S}$ corresponds to known reflectional Coxeter groups [15, 16]. Two reflections in the planes with normals $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are equivalent to the rotation around the vector $\boldsymbol{n}$ by the doubled angle of $2\widehat{\boldsymbol{S}_1\boldsymbol{S}_2}$ between the reflection planes. According to the theory of Coxeter groups, vectors $\boldsymbol{S}$ are called root vectors (or simply roots) and the whole set of roots is called as the system of roots. In the case of $\boldsymbol{S} = \boldsymbol{C} = \boldsymbol{S}^*$, vectors $\boldsymbol{S}$ fall on a circle of unit radius situated in a wavefront plane. It is evident that $X = TXT^{-1} = TXT$, $N = TNT$ where $T = T^{-1} = 1 - 2\boldsymbol{n} \otimes \boldsymbol{n}$ is the reflection operator in the real ($\boldsymbol{n} = \boldsymbol{n}^*$) plane of wavefront. Diad $\boldsymbol{S} \otimes \boldsymbol{C}$ in operator X is projective $(\boldsymbol{S} \otimes \boldsymbol{C})^2 = \boldsymbol{S} \otimes \boldsymbol{C}$. This diad projects any vector situated from the right on a straight line along $\boldsymbol{S}$. Appropriate matrix operators of reflection in hyperplanes of multidimensional spaces were introduced by Weblen and Young [28] and have been used by many mathematicians [15, 16]. The matrix of affine reflections in the n-dimensional space has the form $Y = 1 - 2A$, $A^2 = A$, $Y^2 = 1$. The quadratic form $\boldsymbol{u}A\boldsymbol{u}$ which determines the metric in the space is invariant under transformation Y.

In the lecture on the principle of least action [29], Feynman said in allegorical form that a particle (photon) 'smells' the neighbouring paths to find out whether they have greater or lesser action. Reflection isometries involved in N for isotropic media allow us to continue this allegory. Correcting the choice the photon combines 'smelling' neighbouring paths with 'looking' into infinite sets of mirrors. These sets are connected with different directions of propagation, including opposite ones. The tracelessness of N is caused by the indefiniteness of the reflection operators.

4. Conclusion

In [25] it was shown how kinematical laws of reflection and transmission of light on the plane boundaries can be derived with the help of the ray involutive operators. Consideration carried out above shows that dynamical laws of reflection and transmission can be formulated in terms of involutive operators (reflectional isometries) too. The matrix representation of the traceless refractive index tensors has typical 'triangular' form. Their non-zero non-diagonal matrix element is the doubled ratio of the Fresnel refraction and transmission coefficients of the boundary for two conditions of the coherent scattering of waves which light up the boundary from half-infinite spaces. These tensors in matrix form turn out to be similar to Pauli matrices.

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